On Error Correction Methods for Acceleration of Convergence

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Abstract — This paper provides a unified theory of the error correction methods for acceleration of convergence in linear solvers used in computational electromagnetism, which include the multigrid and deflation methods, explicit and implicit error correction methods as well as the AV method for eddy current analysis. It is shown that in all these methods the error correction with projection matrices play a crucial role, which are constructed so that their null spaces include the slowly converging numerical errors.

I. INTRODUCTION

The multigrid method decomposes the unknowns of iterative linear solvers such as Jacobi and Gauss-Seidel methods into the fast and slowly converging components by use of the restriction-prolongation matrix [1]. The former components with oscillating profiles can be obtained by solving the matrix equations with relatively small numbers of iteration. This solution is then corrected with the slowly converging components determined by the solving the residual equation. The deflation method, which has recently been applied to diffusion and magnetostatic problems [2, 3] for acceleration of linear solution, also carries out the error correction on the basis of the decomposition where the eigenvectors corresponding to the small eigenvalues of the system matrix stand for the slowly converging components.

In this paper, the methods which are based on the correction to eliminate the slowly converging errors, as in the multigrid and deflation methods, are called the error correction (EC) methods, while the multigrid method, which performs multi-level corrections, belongs also to so called the subspace correction method performing the subspace decomposition in each of which decomposed errors are corrected [4].

In the explicit and implicit error correction (EEC and IEC) methods, which have been inspired from the multigrid method, the error correction is not performed by use of the restriction-prolongation matrix but by the matrices which include vectors by whose linear combination the slowly converging components can be expressed [5]. The similarity between the IEC and the AV (or A-phi) methods has been pointed out in [5], which simultaneously solve the original equation and equation for the error correction. Moreover, recently, it has been shown that the convergence in time-periodic electromagnetic fields can be drastically improved by time-periodic error correction (TP-EEC) method [6].

It has been shown that these EC methods are based on the common mathematical principle used in the matrix deflation [7]. However, it remains unclear why the EC methods work well although the correction matrices are not composed of the eigenvectors of small eigenvalues. Indeed, the TP-EEC method solves non-symmetric matrix whose eigenvectors are, in general, complex. For this reason, we need stronger mathematical framework to explain why the EC methods perform convergence acceleration with general correction matrices. This paper will present a unified formulation of the EC methods to clarify their common mathematical principle. It will be shown that the projection matrix, constructed from the correction matrix, plays a crucial role in the acceleration.

II. FORMULATION OF ERROR CORRECTION

Let us consider a system of linear equations

$$\mathrm{A}\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where $\mathrm{A} \in \Re^{n\times n}$ is a symmetric positive definite matrix. (The last condition can be relaxed to be positive-semi definite with some modification in the following formulation.) The approximate solution $\tilde{\boldsymbol{x}}$ to (1) is then compensated as

$$\tilde{\boldsymbol{x}}_{\mathrm{new}} = \tilde{\boldsymbol{x}} + \mathrm{W}\boldsymbol{p} \tag{2}$$

where W=[$\boldsymbol{w}_1$, $\boldsymbol{w}_2$,...,$\boldsymbol{w}_k$], $\boldsymbol{w}_i$, i=1, 2, ..., k, $k << n$ are independent vectors and $\boldsymbol{p}$ denotes a coefficient vector. Now the residual $\boldsymbol{r}$ for $\tilde{\boldsymbol{x}}_{\mathrm{new}}$ given by

$$\boldsymbol{r} = \boldsymbol{b} - \mathrm{A}\tilde{\boldsymbol{x}} - \mathrm{AW}\boldsymbol{p} \tag{3}$$

is enforced to be orthogonal to the space spanned by $\boldsymbol{w}_i$ to have the equation for $\boldsymbol{p}$ as

$$\mathrm{W}^{\mathrm{t}}\mathrm{AW}\boldsymbol{p} = \mathrm{W}^{\mathrm{t}}(\boldsymbol{b} - \mathrm{A}\tilde{\boldsymbol{x}}) \tag{4}$$

Then the EC is made as follows:

$$\tilde{\boldsymbol{x}}_{\mathrm{new}} = \tilde{\boldsymbol{x}} + \mathrm{W}(\mathrm{W}^{\mathrm{t}}\mathrm{AW})^{-1}(\boldsymbol{b} - \mathrm{A}\tilde{\boldsymbol{x}}) \tag{5}$$

By subtracting the exact solution $\boldsymbol{x}$ satisfying (1) from both sides of (5), we have

$$\boldsymbol{e}_{\mathrm{new}} = \mathrm{P}\boldsymbol{e}, \tag{6}$$

where $\boldsymbol{e}$ denotes numerical error and P is defined by

$$\mathrm{P} = \mathrm{I} - \mathrm{W}(\mathrm{W}^{\mathrm{t}}\mathrm{AW})^{-1}\mathrm{W}^{\mathrm{t}}\mathrm{A}, \tag{7}$$

which is a projection matrix satisfying P^2=P. Moreover, it can be shown that $\boldsymbol{e}$ can be decomposed as

$$\boldsymbol{e} = \boldsymbol{e}_{\mathrm{f}} + \boldsymbol{e}_{\mathrm{s}} \tag{8}$$

where

$$\boldsymbol{e}_{\mathrm{f}} \in \mathrm{Ker}(\mathrm{W}^{\mathrm{t}}\mathrm{A}), \quad \boldsymbol{e}_{\mathrm{s}} \in \mathrm{Range}(\mathrm{W}). \tag{9}$$

By definition, each component satisfies $\mathrm{P}\boldsymbol{e}_{\mathrm{s}} = 0$, $\mathrm{P}\boldsymbol{e}_{\mathrm{f}} = \boldsymbol{e}_{\mathrm{f}}$. Hence, the EC eliminates $\boldsymbol{e}_{\mathrm{s}}$ while $\boldsymbol{e}_{\mathrm{f}}$ remains unchanged, the latter of which is effectively reduced through the numerical solution of (1). The vectors $\boldsymbol{w}_i$ are chosen so that they express the slowly convergence components, that is, the reduction rate in $\boldsymbol{e}_{\mathrm{f}}$ is comparatively slow, whereas $\boldsymbol{e}_{\mathrm{f}}$ reduces to zero after small number of iterations. Because the EC selectively eliminates $\boldsymbol{e}_s$, the convergence is effectively accelerated.

III. UNIFIED DERIVATION OF RELAVANT METHODS

It can be found that the following methods are based on the EC given by (6) with differences in the choice of W. TABLE I summarizes the choice of W for each method.

A. Multigrid Method

In this method, $\boldsymbol{e}_s$ corresponds to the error component with smooth profiles which cannot be effectively reduced by, e.g. the Gauss-Seidel and CG methods. The slow components are shown to be in the range of the restriction matrix R mapping vectors in fine grid to those in coarse grid. In the multigrid, $\boldsymbol{e}_s$ is eliminated by (6) where W=R.

B. Deflation Method

The deflation method is based on the decomposition of the unknown in the form $\boldsymbol{x} = \mathrm{P}\boldsymbol{x} + (\mathrm{I}-\mathrm{P})\boldsymbol{x}$, where the first and second terms represent fast and slowly converging components, respectively. The latter can be obtained from

$$(\mathrm{I}-\mathrm{P})\boldsymbol{x} = \mathrm{W}(\mathrm{W}^{\mathrm{t}}\mathrm{AW})^{-1}\mathrm{W}^{\mathrm{t}}\boldsymbol{b}, \qquad (10)$$

while the former is obtained by solving

$$\mathrm{AP}\boldsymbol{x} = \mathrm{P}^{\mathrm{t}}\boldsymbol{b}, \qquad (11)$$

where the commutative property $\mathrm{AP} = \mathrm{P}^{\mathrm{t}}\mathrm{A}$ is used to derive (11). In the typical deflation method, W is composed of eigenvectors corresponding to the small eigenvalues of A. In this case, AP in (11) is proved to have better conditioning than A [2, 7]. Moreover, by substituting (10) into the second term of the decomposition, we have the same form as (5).

C. AV method for eddy current analysis

The A method, widely used in finite element analysis of eddy current problems, which solves (1) with

$$\mathrm{A} = \mathrm{R}^{\mathrm{t}}\mathrm{NR} + \mathrm{j}\omega\mathrm{S}, \qquad (12)$$

has poor convergence especially when the frequency ω is relatively low, where R is the discrete counterpart of the rot operator, N and S are matrices depending on permeability and conductivity [8]. Moreover, the range of G, the discrete counterpart of the grad operator, which satisfies RG=0, represents the slowly converging components. To improve the convergence, the error correction $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \mathrm{G}\boldsymbol{p}$ is performed, the second term of which is obtained by solving the correction equation as

$$\mathrm{j}\omega\mathrm{G}^{\mathrm{t}}\mathrm{SG}\boldsymbol{p} = \mathrm{G}^{\mathrm{t}}(\boldsymbol{b} - \mathrm{j}\omega\mathrm{S}\boldsymbol{x}). \qquad (13)$$

In the AV method, (13) is simultaneously solved with $\mathrm{A}\tilde{\boldsymbol{x}} = \boldsymbol{b}$ that is

$$(\mathrm{R}^{\mathrm{t}}\mathrm{NR} + \mathrm{j}\omega\mathrm{S})\boldsymbol{x} + \mathrm{j}\omega\mathrm{SG}\boldsymbol{p} = \boldsymbol{b}. \qquad (14)$$

D. EEC and IEC, TP-EEC

In the EEC method, the restriction matrix R used in the multigrid method is replaced by a matrix W whose column vectors $\boldsymbol{w}_i$ represent slowly converging components. In this method, $\mathrm{A}\tilde{\boldsymbol{x}} = \boldsymbol{b} - \mathrm{AW}\boldsymbol{p}$ and (4) are alternatively solved. In each step of this process, the error correction (6) is effectively performed. The IEC method simultaneously solves these two equations for the EEC like the AV method. It is shown that the convergence in the ICCG applied to finite element analysis of magnetostatic problems with thin elements has been drastically improved when $\boldsymbol{w}_i$ represent spatially smooth errors [9].

In TP-EEC method [6], the EEC method has been extended to time-periodic eddy current problems which have long time constant compared to the excitation period. To shorten the computational time until the steady state, the temporally smooth components are decomposed from periodic ones. Then the smooth components are determined by solving (4) for the EC (6).

TABLE I
Typical choices of W

Methods	Typical choice of W=[$\boldsymbol{w}_1$, $\boldsymbol{w}_2$, …,$\boldsymbol{w}_k$]
Multigrid	Restriction matrix
Deflation	$\boldsymbol{w}_i$ is eigenvector of small eigenvalue of A
AV for eddy currents	Gradient matrix G
EEC and IEC	$\boldsymbol{w}_i$ expresses spatially smooth error
TP-EEC	$\boldsymbol{w}_i$ expresses temporally smooth error

IV. CONCLUSIONS

It has been shown that the decomposition (2) and EC (6) are common key techniques in various acceleration methods. In the long version, the effect of the EC will be mathematically and numerically discussed. Moreover, effective method for the choice of W will also be discussed.

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